Isabelle Hurbain's Exam Formular

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Most recent version and some context available on:
http://www.pasithee.fr/articles/examformular.html

1 Logs and stuff

$$\log(xy) = \log(x) + \log(y)$$
$$\log(x^p) = p\log(x)$$
$$\log(1/x) = -\log(x)$$
$$\log_b(x) = \frac{\log_k(x)}{log_k(b)}$$
$$x^{i+j} = x^i \cdot x^j$$
$$(x^i)^j = x^{ij}$$
$$b^x = e^{x \ln b}$$
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

2 Binomial coefficients

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$
$$\binom{n}{k_1, k_2, ..., k_m} = \frac{n!}{k_1!k_2!...k_m!} \text{ (where } \sum_{i=1}^{m} k_i = n)$$
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$
$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$
$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$
$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$$

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \frac{n^k}{k!} \leq \left(\frac{ne}{k}\right)^k$$

$$\binom{2n}{n} \overset{Stirling}{\sim} \frac{4^n}{\sqrt{\pi n}}$$

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{6}$$

$$\binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{24}$$

$$\binom{n}{5} = \frac{n(n-1)(n-2)(n-3)(n-4)}{120}$$

3 Distributions

3.1 Binomial distribution

Number of successes in a sequence of n independent yes/no experiments, each of which yields success with probability p.

$$\Pr[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\mathbf{E}[X] = np$$

$$\text{Var}[X] = np(1-p)$$

3.2 Geometric distribution

Probability distribution of the number X of Bernoulli trials needed to get one success.

$$\Pr[X = k] = (1-p)^{k-1} p$$

$$\mathbf{E}[X] = \frac{1}{p}$$

$$\text{Var}[X] = \frac{1-p}{p^2}$$

4 Usual limits

$$\lim_{x \to +\infty} \left(1 + \frac{k}{x}\right)^{mx} = e^{mk}$$

$$\lim_{x \to +\infty} \left(1 - \frac{1}{x}\right)^{x} = \frac{1}{e}$$

$$\lim_{x \to +\infty} x^\alpha = \begin{cases} +\infty, & \alpha > 0 \\ 0, & \alpha < 0 \end{cases}$$

5 Approximations and bounds

- $\forall x \in \mathbb{R}, 1 - x \leq e^{-x}$
- Around 0, $1 + x \sim e^x$
- $\ln n \leq H_n \leq \ln n + 1$
- $\binom{n}{k} \leq 2^n$
- $\binom{n}{k} \leq \frac{n^k}{k!}$
- $\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$
- $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$
- For $0 \leq x \leq 1$, $(1-x)^n \geq 1 - xn$
- For $0 \leq x \leq 1/2, (1-x) \geq e^{-x-x^2}$

6 Numerical values

n	1	2	3	4	5	6	7	8	9
$n!$	1	2	6	24	120	720	5040	40320	362880
n^3	1	8	27	64	125	216	343	512	729
n^4	1	16	81	256	625	1296	2401	4096	6561
2^n	2	4	8	16	32	64	128	256	512
e^x	2.718	7.389	20.086	54.698	148.413	403.428	1096.63	2980.96	8103.08
$\ln x$	0	0.693	1.098	1.386	1.609	1.791	1.946	2.079	2.197
$\log_2 x$	0	1	1.584	2	2.321	2.585	2.807	3	2.170
$\sqrt{x}$	1	1.414	1.732	2	2.236	2.449	2.646	2.828	3

7 Series

$$\sum_{k=1}^{m} k = \frac{m(m+1)}{2}$$

$$\sum_{k=1}^{n} \frac{1}{k} = H_n$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

$$\sum_{k=1}^{n} z^k = z\frac{1-z^n}{1-z}$$

$$\sum_{k=1}^{\infty} z^k = \frac{z}{1-z}$$

$$\sum_{k=1}^{\infty} kz^k = \frac{z}{(1-z)^2}$$

$$\sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z$$

$$\sum_{k=0}^{\infty} \binom{n}{k} = 2^n$$

$$\sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1}$$

8 Randomized algorithms tools[1]

8.1 Markov

Let X be a non-negative random variable. For all $t > 0$, we have

$$\Pr[X \geq t] \leq \frac{\mathbf{E}[X]}{t}$$

8.2 Chebyshev

Let X be a random variable. For all $t > 0$, we have

$$\Pr[|X - \mathbf{E}[X] \geq t] \leq \frac{\mathrm{Var}[X]}{t^2}$$

8.3 First and second moment method

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables which take non-negative integer values. Then

$$\mathbf{E}[X_n] = o(1) \text{ implies } \Pr[X_n = 0] = 1 - o(1)$$

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables. Then

$$\mathbf{E}[X_n] \neq 0 \text{ (for } n \text{ large enough) and } \mathrm{Var}[X_n] = o(\mathbf{E}[X_n]^2) \text{ implies } \Pr[X_n = 0] = o(1)$$

8.4 "Weak" Chernoff

Let $X_1, ..., X_n$ be independent (or, weaker, negatively associated) Bernouilli-distributed variables with $\Pr[X_i = 1] = p_i$ and $\Pr[X_i = 0] = 1 - p_i$. Then the following inequalities hold for $X = \sum_{i=1}^n X_i$ and $\mu = \mathbf{E}[X] = \sum_{i=1}^n p_i$:

$$\Pr[X \geq (1+\delta)\mu] \leq e^{-\mu\delta^2/3} \text{ for all } 0 < \delta \leq 1$$

$$\Pr[X \leq (1-\delta)\mu] \leq e^{-\mu\delta^2/2} \text{ for all } 0 < \delta \leq 1$$

$$\Pr[|X - \mu| \geq \delta\mu] \leq 2e^{-\mu\delta^2/3} \text{ for all } 0 < \delta \leq 1$$

$$\Pr[X \geq t] \leq 2^{-t} \text{ for } t \geq 2e\mu$$

[1] Formulation of theorems from Pr A. Steger awesome Randomized Algorithm Lecture notes, version of September 21st 2011. Typos are my own ;)

8.5 Azuma

Let $(\Omega, \Pr)$ be the product of N discrete probability spaces $(\Omega_1, \Pr_1), ..., (\Omega, \Pr_N)$, and let $X : \Omega \to \mathbb{R}$ be a random variable with the property that the effect of the i-th coordinate is at most c_i. Then for all $t \geq 0$ we have

$$\Pr[X \geq \mathbf{E}[X] + t] \leq e^{-\frac{t^2}{2\sum_{i=1}^{N} c_i^2}}$$

$$\Pr[X \leq \mathbf{E}[X] - t] \leq e^{-\frac{t^2}{2\sum_{i=1}^{N} c_i^2}}$$

8.6 Janson

Let $(\Omega, \Pr)$ be the product of N discrete probability spaces $(\Omega_1, \Pr_1), ..., (\Omega, \Pr_N)$ and let $X_1, ..., X_m$ be indicator variables and $I_1, ..., I_m$ sets of coordinates such that for all $1 \leq i \leq m$ we have: X_i depends only on coordinates in I_i. Furthermore, let $X := \sum_{i=1}^{m} X_i$. Then we have, with

$$\lambda := \mathbf{E}[X] = \sum_{i=1}^{m} \Pr[X_i = 1]$$

$$\Delta := \sum_{i \neq j, I_1 \cap I_j \neq \emptyset} \Pr[X_i = 1 \wedge X_j = 1]$$

for all $0 \leq t \leq \mathbf{E}[X]$ that

$$\Pr[X \leq \mathbf{E}[X] - t] \leq e^{\frac{-t^2}{2(\lambda + \Delta)}}$$

In particular:

$$\Pr[X = 0] \leq e^{-\frac{\lambda^2}{2\lambda + \delta)}} \leq e^{-\min\{\lambda, \lambda^2/\Delta\}/4}$$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40
3	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60
4	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64	68	72	76	80
5	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
6	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108	114	120
7	7	14	21	28	35	42	49	56	63	70	77	84	91	98	105	112	119	126	133	140
8	8	16	24	32	40	48	56	64	72	80	88	96	104	112	120	128	136	144	152	160
9	9	18	27	36	45	54	63	72	81	90	99	108	117	126	135	144	153	162	171	180
10	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190	200
11	11	22	33	44	55	66	77	88	99	110	121	132	143	154	165	176	187	198	209	220
12	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216	228	240
13	13	26	39	52	65	78	91	104	117	130	143	156	169	182	195	208	221	234	247	260
14	14	28	42	56	70	84	98	112	126	140	154	168	182	196	210	224	238	252	266	280
15	15	30	45	60	75	90	105	120	135	150	165	180	195	210	225	240	255	270	285	300
16	16	32	48	64	80	96	112	128	144	160	176	192	208	224	240	256	272	288	304	320
17	17	34	51	68	85	102	119	136	153	170	187	204	221	238	255	272	289	306	323	340
18	18	36	54	72	90	108	126	144	162	180	198	216	234	252	270	288	306	324	342	360
19	19	38	57	76	95	114	133	152	171	190	209	228	247	266	285	304	323	342	361	380
20	20	40	60	80	100	120	140	160	180	200	220	240	260	280	300	320	340	360	380	400